Journal of Engineering Mathematics **24**: 343–368, 1990.

Group method analysis of unsteady free-convective laminar boundary-layer flow on a nonisothermal vertical flat plate

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Received 16 November 1989; accepted in revised form 7 May 1990

Abstract. The transformation group theoretic approach is applied to present an analysis of the problem of unsteady laminar free convection from a non-isothermal vertical flat plate. The application of two-parameter groups reduces the number of independent variables by two, and consequently the system of governing partial differential equations with boundary conditions reduces to a system of ordinary differential equations with appropriate boundary conditions. The possible forms of surface-temperature variations with position and time are derived. The ordinary differential equations are solved numerically using a fourth-order Runge–Kutta scheme and the gradient method. The heat-transfer characteristics for finite values of the Prandtl number Pr are presented, as temperature and velocity distributions.

1. Introduction

The problem of two-dimensional laminar free convection about a semi-infinite flat plate in steady-state conditions is a classical problem. This problem has been fully studied by many investigators. Recently, unsteady conditions of motion and heating of bodies in fluids have become increasingly important in certain applications for some engineering fields of aerodynamics and hydrodynamics. Also a natural-convection flow has been generated due to the temperature difference inside plastic greenhouses. Mankabadi [20] in 1988, considered two pumping systems that can utilize a usable power, about 200 W, for pumping underground water for irrigation purposes. Therefore, it becomes necessary to pay more attention to this problem.

Obviously, the introduction of time as the third independent variable in the unsteady problem increases the complexity of the problem. Many attempts were made to find analytical and numerical solutions, applying certain special conditions and using different mathematical approaches. In 1950, Illingworth [17] studied the problem of unsteady laminar flow of gas near an infinite flat plate. He obtained solutions which are available only with Prandtl number unity and under transient conditions of step change in the surface temperature. The problem of transient free convection at the heated surface has been studied extensively. In 1958, Siegel [29] obtained a solution of this problem using the method of characteristics. The same problem was treated by many investigators: in 1961, by Gebhart [14] using an integral method, in 1962, by Hellums and Churchill [16] and, in 1976, by Callahan and Marner [10]. The latter authors dealt with the same problem with mass transfer and used the finite-difference method to solve the governing equations as an initial-value problem in three independent variables. In 1962, Menold and Yang [21] presented general asymptotic solutions for the same problem for a certain class of surface temperature variations.

In 1969, Heinisch et al. [15], using an integral technique, obtained a system of partial differential equations in two independent variables. This resultant system was again reduced to a system of ordinary differential equations by two separate methods. The first was the method of integral relations, the second was an explicit finite-difference scheme. Approximate temperature and velocity distributions were obtained by both methods. Finally, in 1987, Williams et al. [32] obtained semisimilar solutions for the unsteady free-convective boundary-layer flow on a vertical flat plate. They used an implicit finite-difference method. Solutions were obtained for a number of possible surface-temperature variations with time and position. One will find attractive discussions of the subject in Brindley [8], Burmeister [9], Mahnor [19], Mitchell [23] and Yang [33].

The mathematical technique used in the present analysis is the two-parameter group transformation, which leads to a similarity representation of the problem. The fundamental simplicity and power of this method are well known. Morgan [28], in 1952, presented a theory which led to improvements over earlier similarity methods. In 1952, Michal [22] extended Morgan's theory. Group methods, as a class of methods which lead to a reduction of the number of independent variables, were first introduced by Birkhoff [4, 5]. He made use of one-parameter group transformations to reduce a system of partial differential equations in two independent variables to a system of ordinary differential equations in one independent variable, the similarity variable.

Moran and Gaggioli [13, 26], in 1966 and 1968, presented a general systematic group formalism for similarity analysis. They utilized elementary group theory for the purpose of reducing a given system of partial differential equations to a system of ordinary differential equations in a single variable. Similarity analysis has been applied intensively by Gabbert [12]. For additional discussions on group transformations, one consults Ames [1–3], Eisenhart, Bluman and Cole [6], Boisvert et al. [7], Moran and Gaggioli [24, 25] and [27].

In this work we present a general procedure for reducing the number of independent variables in the governing equations from three to only one independent variable. The used technique is the two-parameter group transformation which is applied to both the governing partial differential equations and the boundary conditions to assure the invariance conditions. The resultant system of ordinary differential equations and appropriate boundary conditions is then solved numerically using a fourth-order Runge–Kutta scheme and the gradient method given in Zettl [35]. Of course, not all surface-temperature variations will lead to a reduction to ordinary differential equations. Therefore, the only possible cases are considered.

2. Formulation of the problem and the governing equations

Consider a natural-convective, laminar, boundary layer adjacent to a semi-infinite, vertical flat plate. The plate is nonisothermal and is heated in an unsteady manner, consequently the temperature distribution over the plate, T_w^*, will be a function of the vertical distance x, and the time t. The fluid is isothermal of constant temperature T_∞^*, far from the plate, such that $T_w^* > T_\infty^*$, Fig. 1.

If we take L as some arbitrary reference length, L/U as a typical time, where $U = \{g\beta L(T_{\text{ref}}^* - T_\infty^*)\}^{1/2}$ is a typical velocity with g the acceleration due to gravity, β is the

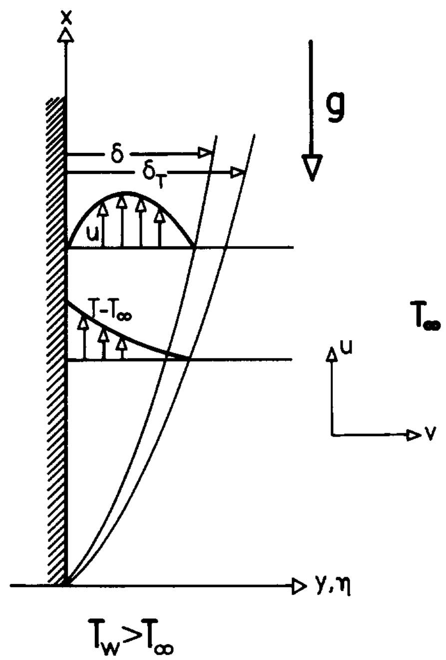

Fig. 1. Physical model of unsteady laminar boundary layer in free convection on a hot vertical flat plate.

volumetric coefficient of thermal expansion and T^*_{ref} is some arbitrary reference temperature, along with the application of the Boussinesq and boundary-layer approximation, the equations of motion may be written as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 , \tag{2.1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = T + \frac{\partial^2 u}{\partial y^2} , \tag{2.2}$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{1}{\mathrm{Pr}}\frac{\partial^2 T}{\partial y^2} , \tag{2.3}$$

with the boundary conditions

$$\begin{aligned} &u = 0 , \quad v = 0 , \quad T = T_w(x, t) \quad \text{at } y = 0 , \\ &u = 0 , \quad T = 0 \quad \text{as } y \to +\infty , \end{aligned} \tag{2.4}$$

where $x = x^*/L$, $y = y^*(\mathrm{Gr})^{1/4}/L$, $u = u^*/U$, $v = v^*(\mathrm{Gr})^{1/4}/U$, $\mathrm{Gr} = g\beta L^3(T^*_{ref} - T^*_\infty)/\nu^2$ is the Grashof number, ν is the kinematic viscosity, $\mathrm{Pr} = \nu/\alpha$ is the Prandtl number, and α is the thermal diffusivity.

From the continuity equation (2.1), there exists a non-dimensional stream function $\psi(x, y, t)$ such that

$$u = \frac{\partial \psi}{\partial y} , \qquad v = -\frac{\partial \psi}{\partial x} ,$$

which satisfies equation (2.1) identically.

If we introduce the non-dimensional temperature defined by

$$\theta = T/T_w ,$$

equations (2.2) and (2.3) become

$$\frac{\partial^2 \psi}{\partial y\, \partial t} + \frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial y\, \partial x} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} = \theta T_w + \frac{\partial^3 \psi}{\partial y^3} , \tag{2.5}$$

$$\left(T_w \frac{\partial \theta}{\partial t} + \theta \frac{\partial T_w}{\partial t}\right) + \frac{\partial \psi}{\partial y}\left(T_w \frac{\partial \theta}{\partial x} + \theta \frac{\partial T_w}{\partial x}\right) - T_w \frac{\partial \psi}{\partial x} \frac{\partial \theta}{\partial y} = \frac{1}{\mathrm{Pr}} T_w \frac{\partial^2 \theta}{\partial y^2} , \tag{2.6}$$

with boundary conditions

$$\begin{aligned} &\frac{\partial \psi}{\partial y}(x, 0, t) = \frac{\partial \psi}{\partial x}(x, 0, t) = 0 , \qquad \theta(x, 0, t) = 1 , \\ &\lim_{y \to \infty} \frac{\partial \psi}{\partial y}(x, y, t) = 0 , \qquad \lim_{y \to \infty} \theta(x, y, t) = 0 . \end{aligned} \tag{2.7}$$

3. Solution of the problem

The method of solution depends on the application of a two-parameter group transformation to the system of partial differential equations (2.5) and (2.6). Under this transformation the three independent variables will be reduced by two and the system of equations transforms into a system of ordinary differential equations in only one independent variable, which is the similarity variable.

3.1. *The group systematic formulation*

The procedure is initiated with the group G, a class of transformation groups of two-parameters (a_1, a_2) of the form

$$G\colon \bar{s} = C^s(a_1, a_2) s + K^s(a_1, a_2) , \tag{3.1}$$

where s stands for $x, y, t, \psi, T_w, \theta$ and the C's and K's are real-valued and at least differentiable in each real argument.

3.2. *The invariance analysis*

To transform the differential equations, transformations of the derivatives are obtained from G via chain-rule operations:

$$\left.\begin{aligned} \bar{s}_{\bar{i}} &= (C^s/C^i) s_i \\ \bar{s}_{\bar{i}\bar{j}} &= (C^s/C^i C^j) s_{ij} \\ \bar{s}_{\bar{i}\bar{j}\bar{k}} &= (C^s/C^i C^j C^k) s_{ijk} \end{aligned}\right\} \quad i = x, y, t; \quad j = x, y, t; \quad \text{and} \quad k = x, y, t \tag{3.2}$$

where s stands for ψ, T_w and θ.

Equation (2.5) is said to be invariantly transformed whenever

$$\bar{\psi}_{\bar{y}\bar{t}} + \bar{\psi}_{\bar{y}}\bar{\psi}_{\bar{y}\bar{x}} - \bar{\psi}_{\bar{x}}\bar{\psi}_{\bar{y}\bar{y}} - \bar{\theta}\bar{T}_{\bar{w}} - \bar{\psi}_{\bar{y}\bar{y}\bar{y}} = H_1(a_1, a_2)[\psi_{yt} + \psi_y\psi_{yx} - \psi_x\psi_{yy} - \theta T_w - \psi_{yyy}], \quad (3.3)$$

for some function $H_1(a_1, a_2)$ which may be a constant.

Substitution from equations (3.1) into equation (3.3) for the independent variables, the functions and their partial derivatives yields

$$\begin{aligned}&[C^{\psi}/C^{y}C^{t}]\psi_{yt} + [(C^{\psi})^2/(C^{y})^2C^{x}]\psi_y\psi_{yx} - [(C^{\psi})^2/(C^{y})^2C^{x}]\psi_x\psi_{yy}\\ &\quad - [C^{\theta}C^{T_w}]\theta T_w - [C^{\psi}/(C^{y})^3]\psi_{yyy} - R_1\\ &\quad = H_1(a_1, a_2)[\psi_{yt} + \psi_y\psi_{yx} - \psi_x\psi_{yy} - \theta T_w - \psi_{yyy}], \end{aligned} \quad (3.4)$$

where

$$R_1 = [C^{\theta}K^{T_w}]\theta + [C^{T_w}K^{\theta}]T_w .$$

The invariance of (3.4) implies $R_1 \equiv 0$. This is satisfied by putting

$$K^{\theta} = K^{T_w} = 0 \quad (3.5)$$

and

$$[C^{\psi}/C^{y}C^{t}] = [(C^{\psi})^2/(C^{y})^2C^{x}] = [C^{\theta}C^{T_w}] = [C^{\psi}/(C^{y})^3] = H_1(a_1, a_2) . \quad (3.6)$$

In a similar manner the invariant transformation of (2.6) gives

$$\begin{aligned}&[C^{\theta}C^{T_w}/C^{t}][T_w\theta_t + \theta(T_w)_t] + [C^{T_w}C^{\theta}C^{\psi}/C^{y}C^{x}][T_w\psi_y\theta_x + \theta\psi_y(T_w)_x - T_w\psi_x\theta_y]\\ &\quad - \frac{1}{\mathrm{Pr}}[C^{T_w}C^{\theta}/(C^{y})^2]T_w\theta_{yy} - R_2 = H_2(a_1, a_2)[T_w\theta_t + \theta(T_w)_t + T_w\psi_y\theta_x\\ &\quad + \theta\psi_y(T_w)_x - T_w\psi_x\theta_y - \frac{1}{\mathrm{Pr}}T_w\theta_{yy}], \end{aligned} \quad (3.7)$$

where

$$\begin{aligned}R_2 &= [K^{T_w}C^{\theta}/C^{t}]\theta_t + [K^{\theta}C^{T_w}/C^{t}](T_w)_t + [K^{T_w}C^{\theta}C^{\psi}/C^{y}C^{x}]\psi_y\theta_x\\ &\quad + [K^{\theta}C^{\psi}C^{T_w}/C^{y}C^{x}]\psi_y(T_w)_x - [K^{T_w}C^{\psi}C^{\theta}/C^{y}C^{x}]\psi_x\theta_y\\ &\quad - \frac{1}{\mathrm{Pr}}[K^{T_w}C^{\theta}/(C^{y})^2]\theta_{yy} . \end{aligned} \quad (3.8)$$

For invariability, we should have

$$[C^{T_w}C^{\theta}/C^{t}] = [C^{T_w}C^{\psi}C^{\theta}/C^{y}C^{x}] = [C^{T_w}C^{\theta}/(C^{y})^2] \equiv H_2(a_1, a_2) , \quad (3.9)$$

and $R_2 \equiv 0$, which is satisfied in accordance with (3.5).

Moreover, the boundary conditions (2.7) are also invariant in form, whenever the condition $K^y = 0$ is appended to (3.5), (3.6) and (3.9). It is obvious that when $K^y = 0$, the transformation of $\theta(x, 0, t) = 1$ implies that $\bar{\theta}(\bar{x}, 0, \bar{t}) = 1$, which is only satisfied if

$$C^\theta = 1 . \tag{3.10}$$

Combining equations (3.6) and (3.9) and invoking the result (3.10), we get

$$C^x = C^y C^\psi , \qquad C^t = (C^y)^2 , \qquad C^{T_w} = C^\psi/(C^y)^3 . \tag{3.11}$$

Finally, we get the two-parameter group G_1 which transforms invariantly the differential equations (2.5), (2.6) and the boundary conditions (2.7). The group G_1 is of the form

$$G_1 : \begin{cases} G_s \begin{cases} \bar{x} = [C^y C^\psi] x + K^x \\ \bar{y} = [C^y] y \\ \bar{t} = [C^y]^2 t + K^t \end{cases} \\ \bar{\psi} = [C^\psi]\psi + K^\psi \\ \bar{T}_w = [C^\psi/(C^y)^3] T_w \\ \bar{\theta} = \theta \end{cases} \tag{3.12}$$

3.3. *The complete set of absolute invariants*

Our aim is to make use of group methods to represent the problem in the form of a system of ordinary differential equations (similarity representation) in a single independent variable (similarity variable). Then we have to proceed in our analysis to obtain a complete set of absolute invariants. In addition to the absolute invariants of the independent variables, there are three absolute invariants corresponding to the three dependent variables ψ, T_w and θ.

If $\eta \equiv \eta(x, y, t)$ is the absolute invariant of the independent variables, then

$$g_j(x, y, t, \psi, T_w, \theta) = F_j(\eta(x, y, t)) , \quad j = 1, 2, 3 ,$$

are the three absolute invariants corresponding to ψ, T_w and θ. The application of a basic theorem in group theory, see [27], states that: a function $g(x, y, t, \psi, \theta, T_w)$ is an absolute invariant of a two-parameter group if it satisfies the two first-order linear differential equations:

$$\begin{aligned} &(\alpha_1 x + \alpha_2) \frac{\partial g}{\partial x} + (\alpha_3 y + \alpha_4) \frac{\partial g}{\partial y} + (\alpha_5 t + \alpha_6) \frac{\partial g}{\partial t} + (\alpha_7 \psi + \alpha_8) \frac{\partial g}{\partial \psi} \\ &\quad + (\alpha_9 \theta + \alpha_{10}) \frac{\partial g}{\partial \theta} + (\alpha_{11} T_w + \alpha_{12}) \frac{\partial g}{\partial T_w} = 0 , \\ &(\beta_1 x + \beta_2) \frac{\partial g}{\partial x} + (\beta_3 y + \beta_4) \frac{\partial g}{\partial y} + (\beta_5 t + \beta_6) \frac{\partial g}{\partial t} + (\beta_7 \psi + \beta_8) \frac{\partial g}{\partial \psi} \\ &\quad + (\beta_9 \theta + \beta_{10}) \frac{\partial g}{\partial \theta} + (\beta_{11} T_w + \beta_{12}) \frac{\partial g}{\partial T_w} = 0 , \end{aligned} \tag{3.13}$$

where

$$\alpha_1 \equiv (\partial C^x/\partial a_1)(a_1^0, a_2^0)\,, \qquad \beta_1 \equiv (\partial C^x/\partial a_2)(a_1^0, a_2^0)\,,$$
$$\alpha_2 \equiv (\partial K^x/\partial a_1)(a_1^0, a_2^0)\ , \quad \text{etc.}\,;$$

(a_1^0, a_2^0) denotes the value of (a_1, a_2) which yields the identity element of the group.

At first, we seek the absolute invariants of the independent variables. Owing to equations (3.13), $\eta(x, y, t)$ is an absolute invariant if it satisfies the two first-order partial differential equations

$$\begin{aligned} &(\alpha_1 x + \alpha_2)\frac{\partial \eta}{\partial x} + \alpha_3 y \frac{\partial \eta}{\partial y} + (\alpha_5 t + \alpha_6)\frac{\partial \eta}{\partial t} = 0\,,\\ &(\beta_1 x + \beta_2)\frac{\partial \eta}{\partial x} + \beta_3 y \frac{\partial \eta}{\partial y} + (\beta_5 t + \beta_6)\frac{\partial \eta}{\partial t} = 0\,, \end{aligned} \tag{3.14}$$

where $\alpha_4 = \beta_4 = 0$, since $K^y = 0$.

Any particular group G' possesses a characteristic set of α's and β's; and consequently a characteristic set of absolute invariants which are yielded by (3.13).

For the two-parameter group G_s there is one and only one functionally independent solution to (3.14), i.e., the coefficient matrix of $\{\partial\eta/\partial x, \partial\eta/\partial y, \partial\eta/\partial t\}$ must have rank two. The matrix has rank two whenever at least one of its two-by-two submatrices has a non-vanishing determinant. This condition is met with whenever at least one of the following conditions is satisfied

$$\lambda_{31}x + \lambda_{32} \neq 0\,, \qquad \lambda_{35}t + \lambda_{36} \neq 0\,, \qquad \lambda_{15}xt + \lambda_{16}x + \lambda_{25}t + \lambda_{26} \neq 0\,, \tag{3.15}$$

where

$$\lambda_{ij} \equiv \alpha_i\beta_j - \alpha_j\beta_i\,, \qquad (i, j = 1, 2, 3, 5, 6)\,.$$

The system (3.14) can be rewritten in terms of the λ's:

$$\begin{aligned} &(\lambda_{31}x + \lambda_{32})\frac{\partial \eta}{\partial x} + (\lambda_{35}t + \lambda_{36})\frac{\partial \eta}{\partial t} = 0\,,\\ &(\lambda_{31}x + \lambda_{32})\, y\frac{\partial \eta}{\partial y} - (\lambda_{15}xt + \lambda_{16}x + \lambda_{25}t + \lambda_{26})\frac{\partial \eta}{\partial t} = 0\,. \end{aligned} \tag{3.16}$$

Referring to the transformation group G_1 given by (3.9), and making use of the definition of the α's and β's and invoking that $\alpha_5 = 2\alpha_3$, $\beta_5 = 2\beta_3$, we get

$$\lambda_{35} = \alpha_3\beta_5 - \alpha_5\beta_3 = 0\,. \tag{3.17}$$

According to the conditions (3.15), three main cases arise:

Case (1): none of the coefficients in (3.16) vanishes.

Subcase (1-a): $\lambda_{31} = 0$, $\lambda_{35} = 0$, $\lambda_{32} \neq 0$, $\lambda_{36} \neq 0$.

Subcase (1-b): $\lambda_{31} \neq 0$, $\lambda_{35} = 0$, $\lambda_{36} \neq 0$.

Subcase (1-c): $\lambda_{31} \neq 0$, $\lambda_{35} \neq 0$.

Subcase (1-d): $\lambda_{31} = 0$, $\lambda_{35} \neq 0$, $\lambda_{32} \neq 0$.

Invoking the result of (3.17), the cases for which $\lambda_{35} \neq 0$ are not considered.

Case (2): only one of the coefficients in (3.16) vanishes identically.

Subcase (2-a): $(\lambda_{31}x + \lambda_{32}) \equiv 0$, $(\lambda_{35}t + \lambda_{36}) \not\equiv 0$,

$$(\lambda_{15}xt + \lambda_{16}x + \lambda_{25}t + \lambda_{26}) \not\equiv 0 .$$

The result obtained from equation (3.16) corresponding to this case is $\partial\eta/\partial t = 0$. In fact, this is the case representing the steady-state conditions.

Subcase (2-b): $(\lambda_{31}x + \lambda_{32}) \not\equiv 0$, $(\lambda_{35}t + \lambda_{36}) \equiv 0$,

$$(\lambda_{15}xt + \lambda_{16}x + \lambda_{25}t + \lambda_{26}) \not\equiv 0 .$$

From equation (3.16) this yields a solution $\eta = \eta(y, t)$.

Subcase (2-c): $(\lambda_{31}x + \lambda_{32}) \not\equiv 0$, $(\lambda_{35}t + \lambda_{36}) \not\equiv 0$,

$$(\lambda_{15}xt + \lambda_{16}x + \lambda_{25}t + l_{26}) \equiv 0 .$$

From equation (3.16), it can be proved that the vanishing of $(\lambda_{15}xt + \lambda_{16}x + \lambda_{25}t + \lambda_{26})$ yields a solution $\eta = \eta(x, t)$, i.e., independent of y; this is unacceptable in view of the boundary conditions.

Case (3): two of the coefficients in (3.16) vanish identically. As pointed out above, any case corresponding to the vanishing of $(\lambda_{15}xt + \lambda_{16}x + \lambda_{25}t + \lambda_{26})$ will not be considered. In this case we may have $(\lambda_{31}x + \lambda_{32}) \equiv 0$, $(\lambda_{35}t + \lambda_{36}) \equiv 0$, $(\lambda_{15}xt + \lambda_{16}x + \lambda_{25}t + \lambda_{26}) \not\equiv 0$. Making use of the definition of the λ's in (3.15), and invoking that $\lambda_{35} = 0$, this case corresponds to $\lambda_{31} = \lambda_{32} = \lambda_{35} = \lambda_{36} = 0$, which implies that $\lambda_{15} = \lambda_{25} = \lambda_{16} = \lambda_{26} = 0$, which reduces to the case of vanishing of $(\lambda_{15}xt + \lambda_{16}x + \lambda_{25}t + \lambda_{26})$, which is again unacceptable.

Now, our attention will be focussed on those cases which are consistent with the characteristics of the group and with the boundary conditions.

Case (1)

Subcase (1-a): In this case the following general procedure is utilized. According to a well-known standard technique for linear partial differential equations, the first equation of (3.16) has the general solution

$$\eta = f(y, \xi(x, t)) , \tag{3.18}$$

where f is arbitrary and ξ is any function such that $\xi(x, t) = \text{constant}$ provides a solution to

$$\frac{\mathrm{d}x}{\lambda_{32}} = \frac{\mathrm{d}t}{\lambda_{36}} . \tag{3.19}$$

The solution of this equation gives

$$\xi(x, t) = \lambda_{36}x - \lambda_{32}t = \text{constant} . \tag{3.20}$$

However, to obtain a solution to the first equation of (3.16) it is also necessary, of course, to satisfy the second equation of (3.16). Thus, with (3.18), the second equation of (3.16) becomes

$$y\frac{\partial f}{\partial y} - \left(\frac{\lambda_{15}xt + \lambda_{16}x + \lambda_{25}t + \lambda_{26}}{\lambda_{31}x + \lambda_{32}} \frac{\partial \xi}{\partial t}\right)\frac{\partial f}{\partial \xi} = 0 . \tag{3.21}$$

Inasmuch as ξ is independent of y, the coefficient of $\partial f/\partial \xi$,

$$\left(\frac{\lambda_{15}xt + \lambda_{16}x + \lambda_{25}t + \lambda_{26}}{\lambda_{31}x + \lambda_{32}}\right)\frac{\partial \xi}{\partial t} , \tag{3.22}$$

is also independent of y. Thus for f to be a function of y and ξ, it is necessary for the coefficient to depend only on ξ, i.e., equation (3.21) can be rewritten as

$$y\frac{\partial f}{\partial y} - h(\xi)\frac{\partial f}{\partial \xi} = 0 . \tag{3.23}$$

where h is given by (3.22).

Now, we are seeking solutions of equation (3.23) and consequently of equations (3.16) in the form

$$f = \phi(yH(\xi)) , \tag{3.24}$$

where $H(\xi)$ is given by the ordinary differential equation

$$h(\xi)\frac{\mathrm{d}\ln H}{\mathrm{d}\xi} = 1 , \tag{3.25}$$

obtained via substitution of (3.24) into (3.23). The solution of (3.25) gives

$$H(\xi) = \exp\left(\int \frac{\mathrm{d}\xi}{h(\xi)}\right) . \tag{3.26}$$

Substituting from (3.20) into the expression of h given by equation (3.22) and using $\lambda_{31} = 0$, we get

$$h = -(\lambda_{15}xt + \lambda_{16}x + \lambda_{25}t + \lambda_{26}) . \tag{3.27}$$

Since h is determined by ξ alone,

$$0 \equiv \left.\frac{\partial h}{\partial x}\right]_{\xi} = \left.\frac{\partial h}{\partial x}\right]_{t} + \left.\frac{\partial h}{\partial t}\right]_{x}\left.\frac{\partial t}{\partial x}\right]_{\xi} . \tag{3.28}$$

With equations (3.20) and (3.27), equation (3.28) becomes

$$(\lambda_{15}t + \lambda_{16}) + (\lambda_{15}x + \lambda_{25})\,\frac{\lambda_{36}}{\lambda_{32}} = 0\,. \tag{3.29}$$

The conditions necessary for (3.29) to be satisfied are $\lambda_{15} = 0$, and $\lambda_{36}\lambda_{25} = -\lambda_{16}\lambda_{32}$. These can easily be satisfied, since the α's and β's and consequently the λ's can be chosen arbitrarily. Thus,

$$h(\xi) = -\left(\frac{\lambda_{16}}{\lambda_{36}}\,\xi + \lambda_{26}\right). \tag{3.30}$$

Then, with (3.26),

$$H(\xi) = \exp\left(-\int \frac{\mathrm{d}\xi}{(\lambda_{16}/\lambda_{36})\xi + \lambda_{26}}\right). \tag{3.31}$$

Integration of (3.31) yields

$$H(\xi) \sim (\lambda_{16}x + \lambda_{25}t + \lambda_{26})^{-\lambda_{36}/\lambda_{16}}\,, \tag{3.32}$$

provided $\lambda_{16} \neq 0$. For the case of vanishing λ_{16}, equation (3.31) gives another solution. The characteristics of the present transformation group in the case of vanishing λ_{16} imply that $\lambda_{36} = 0$, which contradicts the conditions of the present case. The absolute invariant η can be obtained from equations (3.18) and (3.24) as

$$\eta = \phi(y(Ax + Bt + C)^{-1/2})\,, \tag{3.33}$$

where the constants A, B and C stand for λ_{16}, λ_{25} and λ_{26} respectively, and the exponent $(-\lambda_{36}/\lambda_{16}) = -\frac{1}{2}$.

Without loss of generality, the function ϕ can be taken to be the identity function. Thus,

$$\eta = y\pi_1(x, t)\,, \tag{3.34}$$

where

$$\pi_1(x, t) = (Ax + Bt + C)^{-1/2}\,. \tag{3.35}$$

Subcase (1-b):

When dealing with this case and following the same procedure, it is found that the characteristics of the group of transformations can not satisfy the conditions of this subcase.

Case (2)

Subcase (2-b):

Applying the conditions of this case to equations (3.16), we get $\partial\eta/\partial x = 0$, and hence (3.14) reduces to

$$\begin{aligned} \alpha_3 y \frac{\partial \eta}{\partial y} + (\alpha_5 t + \alpha_6) \frac{\partial \eta}{\partial t} &= 0 , \\ \beta_3 y \frac{\partial \eta}{\partial y} + (\beta_5 t + \beta_6) \frac{\partial \eta}{\partial t} &= 0 . \end{aligned} \tag{3.36}$$

Using the standard technique for linear partial differential equations, $\eta \equiv \eta(y, t)$ can be obtained as a solution of (3.36) of the form

$$\eta = y\pi_2(t) , \tag{3.37}$$

where

$$\pi_2(t) = K(a_1 t + b_1)^{-1/2} , \tag{3.38}$$

$a_1 = \alpha_5 = \beta_5$, and $b_1 = \alpha_6 = \beta_6$ are constraints. The arbitrary constant K may be taken as unity. The exponent is $-\alpha_3/\alpha_5 = -\beta_3/\beta_5 = -\frac{1}{2}$. Then the absolute invariant for this case will be

$$\eta = y(a_1 t + b_1)^{-1/2} . \tag{3.39}$$

In the next step, we have to obtain the absolute invariants corresponding to the dependent variables ψ, T_w and θ. From (3.10), θ is itself an absolute invariant. Thus,

$$g_1(x, y, t; \theta) = \theta(\eta) . \tag{3.40}$$

By observation of (3.13), it is apparent that any function $g_2(x, t; \psi)$ which satisfies

$$\begin{aligned} (\alpha_1 x + \alpha_2) \frac{\partial g_2}{\partial x} + (\alpha_5 t + \alpha_6) \frac{\partial g_2}{\partial t} + (\alpha_7 \psi + \alpha_8) \frac{\partial g_2}{\partial \psi} &= 0 , \\ (\beta_1 x + \beta_2) \frac{\partial g_2}{\partial x} + (\beta_5 t + \beta_6) \frac{\partial g_2}{\partial t} + (\beta_7 \psi + \beta_8) \frac{\partial g_2}{\partial \psi} &= 0 , \end{aligned} \tag{3.41}$$

provides a solution to (3.13). The solution of equations (3.41) gives

$$g_2(x, t; \psi) = \phi_1(\psi/\Gamma(x, t)) = F(\eta) . \tag{3.42}$$

In a similar manner, we get

$$g_3(x, t; T_w) = \phi_2(T_w/\omega(x, t)) = E(\eta) , \tag{3.43}$$

where $\Gamma(x, t)$ and $\omega(x, t)$ are functions to be determined. Without loss of generality, the ϕ's in (3.42) and (3.43) are selected to be the identity functions. Then we can express the functions $\psi(x, y, t)$ and $T_w(x, t)$ in terms of the absolute invariants $F(\eta)$ and $E(\eta)$ in the form

$$\psi(x, y, t) = \Gamma(x, t)F(\eta) , \tag{3.44}$$

$$T_w(x, t) = \omega(x, t)E(\eta) . \tag{3.45}$$

Since $\omega(x, t)$ and $T_w(x, t)$ are independent of y, whereas η depends on y, it follows that E in (3.45) must be equal to a constant. Then (3.45) becomes

$$T_w(x, t) = T_0\omega(x, t) . \tag{3.46}$$

The forms of the functions $\Gamma(x, t)$ and $\omega(x, t)$ in (3.44) and (3.46) respectively, are those for which the governing equations (2.5) and (2.6) reduce to ordinary differential equations.

4. The reduction to ordinary differential equations

As the general analysis proceeds, the established forms of the dependent and independent absolute invariants are used to obtain ordinary differential equations. Generally, the absolute invariant $\eta(x, y, t)$ has the form

$$\eta = y\pi(x, t) , \tag{4.1}$$

where the function π will be assigned its own form corresponding to each specified case.

Substitution from (3.40), (3.44) and (3.46) into equation (2.5) yields, after dividing by $\Gamma\pi^3$ and rearranging the terms,

$$F''' + \left(\frac{1}{\pi}\frac{\partial\Gamma}{\partial x}\right)(FF'' - F'^2) - \left(\frac{\Gamma}{\pi^2}\frac{\partial\pi}{\partial x}\right)F'^2 - \left(\frac{1}{\Gamma\pi^2}\frac{\partial\Gamma}{\partial t}\right)F' - \left(\frac{1}{\pi^3}\frac{\partial\pi}{\partial t}\right)(\eta F'' + F') + \left(\frac{T_0\theta}{\Gamma\pi^3}\right)\theta = 0 , \tag{4.2}$$

where the primes refer to differentiation with respect to η.

Inasmuch as the first term of (4.2) has the coefficient 1, for (4.2) to reduce to an expression in the single independent variant η, it is necessary that the remaining coefficients be constants or functions of η alone. Thus, since π, Γ and ω are independent of y,

$$\frac{1}{\pi}\frac{\partial\Gamma}{\partial x} = C_1 , \tag{4.3}$$

$$\frac{\Gamma}{\pi_2}\frac{\partial\pi}{\partial x} = C_2 , \tag{4.4}$$

$$\frac{1}{\Gamma\pi^2}\frac{\partial\Gamma}{\partial t} = C_3 , \tag{4.5}$$

$$\frac{1}{\pi^3}\frac{\partial\pi}{\partial t} = C_4 , \tag{4.6}$$

$$\frac{T_0\omega}{\Gamma\pi^3} = C_5 , \tag{4.7}$$

where the C's are constants to be determined for each individual case corresponding to each set of absolute invariants. It follows, then, that (4.2) may be rewritten as

$$F''' + (C_1F - C_4\eta)F'' - (C_1 + C_2)F'^2 - (C_3 + C_4)F' + C_5\theta = 0 . \tag{4.8}$$

Following the same procedure, substitution of ψ, θ, T_w and their partial derivatives in terms of F, θ, ω and η into equation (2.6) yields

$$\frac{1}{\Pr}\theta'' + (C_1F - C_4\eta)\theta' - (C_6F' + C_7)\theta = 0\,, \tag{4.9}$$

where the constants C_6 and C_7 are given by

$$C_6 = \frac{\Gamma}{\pi\omega}\frac{\partial\omega}{\partial x}\,, \tag{4.10}$$

$$C_7 = \frac{1}{\omega\pi^2}\frac{\partial\omega}{\partial t}\,. \tag{4.11}$$

The coupled nonlinear ordinary differential equations (4.8) and (4.9) with the following boundary conditions:

$$\begin{aligned} &F = F' = 0\,, \quad \theta = 1 \quad \text{at } \eta = 0\,, \\ &F' = 0\,, \qquad \theta = 0 \quad \text{as } \eta \to \infty\,, \end{aligned} \tag{4.12}$$

are now the new system representing the problem instead of equations (2.5), (2.6) and (2.7).

It remains to utilize each of the η's in turn with (4.3) to (4.7) and (4.10), (4.11) to evaluate the C's appearing in the ordinary differential equations (4.8) and (4.9) and consequently to evaluate the corresponding expressions of the functions Γ and ω.

5. Subcase (1-a): $\eta = y\pi_1(x, t) = y(Ax + Bt + C)^{-1/2}$

For this case, it follows that

$$\frac{\partial\pi_1}{\partial x} = -\frac{A}{2}\pi_1^3\,, \qquad \frac{\partial\pi_1}{\partial t} = -\frac{B}{2}\pi_1^3\,, \tag{5.1}$$

which on substitution into (4.3) to (4.6) yields

$$\Gamma(x, t) = \frac{2C_1}{A}(Ax + Bt + C)^{1/2}\,, \tag{5.2}$$

$$C_1 = -C_2\,, \tag{5.3}$$

$$C_4 = -\frac{B}{2}\,. \tag{5.4}$$

Coupling of equations (4.5) and (5.4) gives

$$C_3 = -C_4 = \frac{B}{2}\,. \tag{5.5}$$

The constant C_5 in equation (4.7) may be taken to be unity. This can be achieved without restricting the expression of T_w. Thus,

$$T_w = \frac{2C_1/A}{(Ax + Bt + C)}, \tag{5.6}$$

which is the possible form of the surface-temperature variation with respect to x and t corresponding to the present case. Substitution for the established expressions for π_1, Γ and ω into (4.10) and (4.11) gives

$$C_6 = -2C_1, \tag{5.7}$$

$$C_7 = -B. \tag{5.8}$$

Substituting the above-obtained values of the constants into equations (4.8) and (4.9), we get

$$F''' + \left(C_1 F + \frac{B}{2}\eta\right) F'' + \theta = 0, \tag{5.9}$$

$$\frac{1}{Pr}\theta'' + \left(C_1 F + \frac{B}{2}\eta\right)\theta' + (2C_1 F' + B)\theta = 0, \tag{5.10}$$

with the boundary conditions given by (4.12).

We scale the unknown constant C_1 out of the equations (5.9) and (5.10) by writing

$$F = C_1^{-3/4}\hat{F}, \qquad \eta = C_1^{-1/4}\hat{\eta}, \qquad B = C_1^{1/2}\hat{B}, \tag{5.11}$$

assuming that $C_1 > 0$ which is the only physically realistic possibility, we get

$$F''' + \left(F + \frac{B}{2}\eta\right) F'' + \theta = 0, \tag{5.12}$$

$$\frac{1}{\mathrm{Pr}}\theta'' + \left(F + \frac{B}{2}\eta\right)\theta' + (2F' + B)\theta = 0. \tag{5.13}$$

Here the "^"s have been discarded, and the appropriate boundary conditions (4.12) are unaltered.

For the above case, the boundary-layer characteristics are:

(i) The vertical velocity

$$u = \frac{\partial\psi}{\partial y} = \frac{2}{A}F'. \tag{5.14}$$

(ii) The horizontal velocity

$$v = -\frac{\partial\psi}{\partial x} = \frac{\eta F' - F}{(Ax + Bt + C)^{1/2}}. \tag{5.15}$$

(iii) The surface heat flux

$$q = \frac{2}{A(Ax + Bt + C)^{3/2}}[-\theta'(0)]. \tag{5.16}$$

For the case of large values of B, write

$$\theta = \theta(\xi), \qquad F = B^{-3/2}\phi(\xi) \quad \text{and} \quad \xi = B^{-1/2}\eta, \tag{5.17}$$

in (5.13) and let $B \to \infty$. Then one obtains

$$\frac{1}{\mathrm{Pr}}\,\theta'' + \frac{\xi}{2}\,\theta' + \theta = 0 \tag{5.18}$$

with the boundary conditions,

$$\theta(0) = 1, \qquad \theta(\infty) = 0 \tag{5.19}$$

In this case a solution of (5.18) satisfying the boundary conditions (5.19) does not exist, which indicates that large values of B should not be taken into consideration.

6. Subcase (2-b): $\eta = y\pi_2(t) = y(a_1 t + b_1)^{-1/2}$

For this case, it follows that

$$\frac{\partial \pi_2}{\partial x} = 0, \qquad \frac{\partial \pi_2}{\partial t} = \frac{-a_1/2}{(a_1 t + b_1)^{3/2}}. \tag{6.1}$$

The same procedure is adopted to evaluate the constants and to deduce the expressions for Γ and ω corresponding to this case.

From (4.4) and (6.1), it is seen that $C_2 = 0$, and integration of equation (4.3) yields

$$\Gamma(x, t) = \frac{(C_1 x + b_2)}{(a_1 t + b_1)^{1/2}}, \tag{6.2}$$

where b_2 is the constant of integration. Substitution of Γ, π_2, $\partial\Gamma/\partial t$ and $\partial\pi_2/\partial t$ into equations (4.5) and (4.6) yields

$$C_3 = C_4 = -\frac{a_1}{2}. \tag{6.3}$$

Assigning the value unity to C_5 in equation (4.7), we get

$$T_0\omega(x, t) = T_w(x, t) = \frac{C_1 x + b_1}{(a_1 t + b_1)^2}, \tag{6.4}$$

which is the possible form for the surface-temperature distributions as a function of x and t corresponding to this case.

In a similar manner, using the above expressions for π_2, Γ and ω in equations (4.10) and (4.11), we get

$$C_6 = C_1 \quad \text{and} \quad C_7 = -2a_1. \tag{6.5}$$

The ordinary differential equations representing this case are now obtainable by substituting the values of the constants $C_1, C_2, \ldots, C_7$ into equations (4.8) and (4.9), which yields

$$F''' + \left(C_1 F + \frac{a_1}{2}\eta\right) F'' - C_1 F'^2 + a_1 F' + \theta = 0\,, \tag{6.6}$$

$$\frac{1}{\mathrm{Pr}}\theta'' + \left(C_1 F + \frac{a_1}{2}\eta\right)\theta' - (C_1 F' - 2a_1)\theta = 0\,, \tag{6.7}$$

with the boundary conditions given by (4.12).

We scale the unknown constant C_1 out of the equations (6.6) and (6.7) by writing

$$F = C_1^{-3/4}\hat{F}\,, \qquad \eta = C_1^{-1/4}\hat{\eta}\,, \qquad a_1 = C_1^{1/2}\hat{a}_1\,, \tag{6.8}$$

assuming that $C_1 > 0$, which is the only physically realistic possibility. We get

$$F''' + \left(F + \frac{a_1}{2}\eta\right) F'' - F'^2 + a_1 F' + \theta = 0\,, \tag{6.9}$$

$$\frac{1}{\mathrm{Pr}}\theta'' + \left(F + \frac{a_1}{2}\eta\right)\theta' - (F' - 2a_1)\theta = 0\,. \tag{6.10}$$

Here, the "^"s have been discarded, and the appropriate boundary conditions (4.12) are unaltered.

The boundary-layer characteristics for this case are:

(i) The vertical velocity

$$u = \frac{x + b_2}{a_1 t + b_1} F'\,. \tag{6.11}$$

(ii) The horizontal velocity

$$v = -\frac{1}{(a_1 t + b_1)^{1/2}} F\,. \tag{6.12}$$

(iii) The surface heat flux

$$q = \frac{x + b_2}{(a_1 t + b_1)^{5/2}} [-\theta'(0)]\,. \tag{6.13}$$

Special case of subcase (2-b):

This case arises as a special case of subcase (2-b) when the asymptotic solution is considered. This situation is valid at large distance x. Therefore, all partial derivatives with respect to x will be neglected. Applying this situation to equations (4.3) and (4.10), we get

$$C_1 = C_6 = 0\,. \tag{6.14}$$

Utilizing equations (4.3) to (4.7), (4.10) and (4.11) yields

$$\Gamma(t) = K_1(a_1 t + b_1)^l , \tag{6.15}$$

where K_1 is the constant of integration, which may be equal to unity and l is a real constant given by

$$l = C_3/a_1 . \tag{6.16}$$

Substitution of (6.15) into (4.7) and equating C_5 to unity yields

$$T_w = (a_1 t + b_1)^r , \tag{6.17}$$

where

$$r = l - \frac{3}{2} = \frac{C_3}{a_1} - \frac{3}{2} . \tag{6.18}$$

Coupling equations (4.11) and (6.17) gives

$$C_7 = r a_1 . \tag{6.19}$$

Substituting the obtained values of the contants $C_1, C_2, \ldots, C_7$ into equations (4.8) and (4.9) respectively, yields

$$F''' + \frac{a_1}{2} \eta F'' - a_1(1 + r)F' + \theta = 0 , \tag{6.20}$$

$$\frac{1}{\mathrm{Pr}} \theta'' + \frac{a_1}{2} \eta\theta' - a_1 r\theta = 0 . \tag{6.21}$$

with the boundary conditions given by (4.12).

We scale the unknown constant a_1 out of the equations (6.20) and (6.21) by writing

$$F = a_1^{-3/2} \hat{F} , \qquad \eta = a_1^{-1/2} \hat{\eta} , \tag{6.22}$$

assuming that $a_1 > 0$, which is the only physically realistic possibility. We get

$$F''' + \tfrac{1}{2}\eta F''' - (1 + r)F' + \theta = 0 , \tag{6.23}$$

$$\frac{1}{\mathrm{Pr}} \theta'' + \tfrac{1}{2}\eta\theta' - r\theta = 0 . \tag{6.24}$$

Here the "^"s have been discarded, and the appropriate boundary conditions (4.12) are unaltered.

Equation (6.17) gives the surface-temperature distribution corresponding to this case, which is independent of x, i.e., uniform. It is a function of time t. The surface temperature may increase or decrease with time according to r being positive or negative, respectively.

As a special case, for $\mathrm{Pr} = 1$, equation (6.24) takes the form

$$\theta'' + \tfrac{1}{2}\eta\theta' - r\theta = 0 , \tag{6.25}$$

of which the analytic solution, in tems of the confluent hypergeometric function, see Slater [30], is

$$\theta(\eta) = \frac{\Gamma(r+1)}{\Gamma(1/2)}\, e^{-\eta^2/4} U\left(r + \frac{1}{2}\,;\, \frac{1}{2}\,;\, \frac{\eta^2}{4}\right), \tag{6.26}$$

where $\Gamma(r)$ is the gamma function.

The boundary-layer characteristics are:

(i) The vertical velocity

$$u = (t + b_1)^{r+1} F' . \tag{6.27}$$

(ii) The horizontal velocity

$$v = 0 , \tag{6.28}$$

which is evident from the continuity equation.

(iii) The surface heat flux

$$q = (t + b_1)^{(r-1)/2}[-\theta'(0)] . \tag{6.29}$$

7. Results and discussion

We shall deal with each case of the three derived subcases individually.

Case (1). Subcase (1 − a)

It was found that no numerical results could be obtained for the system of equations (5.12) and (5.13), with the boundary conditions (4.12), corresponding to the subcase (1-a). This is due to the fact that the boundary value of the temperature derivative, $\theta'(0)$ takes on infinite values. Here, we may invoke the interpretation which has been given by Sparrow and Gregg [31]. In their study of the steady case of temperature variation according to the relation $T_w - T_\infty = Nx^n$, they investigated that for values of $n < -0.6$, $\theta'(0)$ is negative. Physically, this corresponds to a heat transfer from the fluid to the wall. They stated that, for $n < -0.6$, investigation of the mathematical model shows that there is an infinite source of energy in the fluid at the leading edge, which does not, physically, exist. Also the case of large values of B does not lead to any solution, as has been stated in Section 5.

For $n \simeq -1.0$, the tendency for $\theta'(0)$ to be infinite cannot be a physically realistic situation. It could be interpreted from a mathematical point of view. The infinite value of θ' at $\eta = 0$ means that the vertical axis will be an asymptotic line for the temperature profile. This contradicts the boundary condition $\theta(0) = 1$. Therefore, it seems to be impossible to obtain solutions for such cases for which, theoretically speaking, $\theta'(0) \to \infty$.

If the coefficient of t, B, is set to zero, this case reduces to the steady-state case with non-uniform surface temperature varying inversely with x. As has been investigated by Sparrow and Gregg [31], no solutions were obtainable for the case corresponding to $T_w = x^n$, for $n < -0.8$. Also, this case was not considered in the investigation of Williams et al. [32], where they stated that it is of no practical interest. For the same reasons, this case is not investigated here.

Case (2). Subcase (2-b)

Owing to equation (6.4), the variation of T_w with time depends on a_1, the coefficient of t. Figures 2 and 3 show the effect of a_1 on the temperature and velocity profiles respectively. The results are obtained for $\mathrm{Pr} = 0.7$ corresponding to a set of negative values of a_1 (increasing temperature with time) and positive values (decreasing temperature with time).

From Fig. 2 it is noted that the temperature profile overshoots in the region of the boundary layer near the plate. This phenomenon occurs for values of $a_1 > 1$, and becomes stronger as a_1 increases. This means that this phenomenon is accompanied by those cases for which T_w decreases rapidly with time. This phenomenon does not appear for any case corresponding to a negative value of a_1, i.e., increasing temperature with time.

In Fig. 3 the velocity profiles indicate the increase of the boundary-layer thickness corresponding to decreasing values of a_1.

Figure 4 shows the effect of the unsteadiness of T_w on the surface heat flux represented by $-\theta'(0)$. The relations between a_1 and $-\theta'(0)$ are obtained for $\mathrm{Pr} = 0.7$ and 1.0. These relations indicate that the heat flux becomes negative for a_1 near to and greater than unity. At $a_1 = 0$, the value of $-\theta'(0)$ is almost unity for the case of $\mathrm{Pr} = 1$. This corresponds to the steady-state case with non-uniform surface-temperature distribution which varies linearly with x, equation (6.4).

Figure 5 illustrates an upper bound for a_1 as a function of Pr. It is clear that the upper bound of a_1 decreases as the Prandtl number Pr increases.

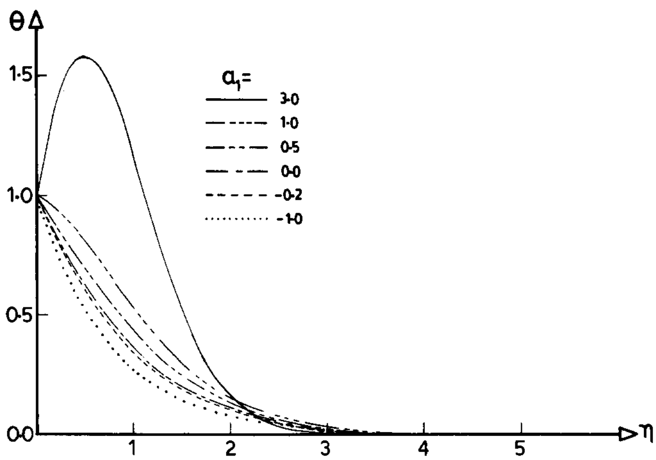

Fig. 2. Calculated dimensionless temperature profiles for fixed $\mathrm{Pr} = 0.7$ and varying values of $T_w = (x + b_2)/(0.4472a_1t + b_1)^2$.

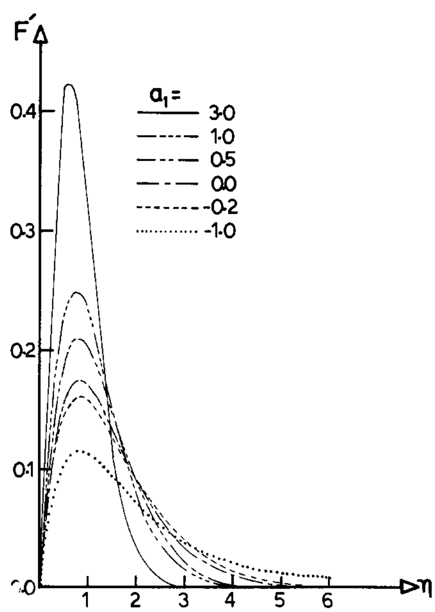

Fig. 3. Calculated dimensionless velocity profiles for fixed $\mathrm{Pr} = 0.7$ and varying values of $T_w = (x + b_2)/(0.4472a_1t + b_1)^2$.

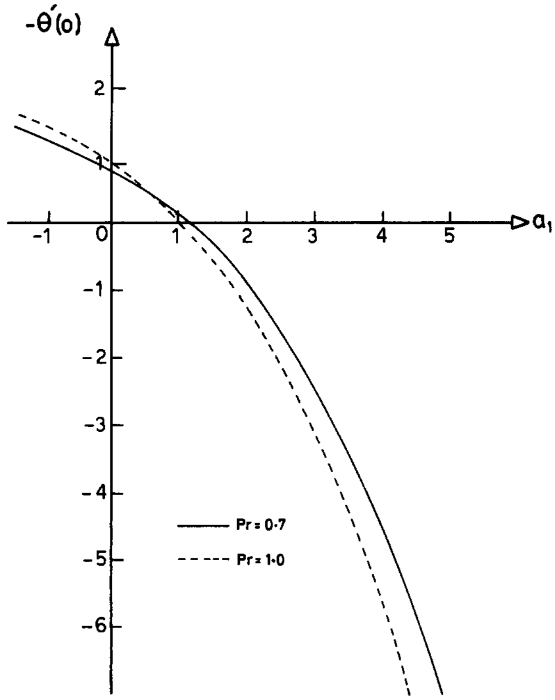

Fig. 4. Plot of the surface heat flux, $-\theta'(0)$, against a_1 for varying values of Pr where $T_w = (x + b_2)/(0.4472a_1t + b_1)^2$.

The effect of Pr on the boundary-layer characteristics is illustrated in Figs. 6 and 7. The results are obtained for $a_1 = 0.4472$ and Pr = 0.7, 2, 6 and 10.

For the temperature profile, Fig. 6 indicates the occurrence of the rapid increase in θ near the plate. This becomes more evident for larger values of Pr. Also, Fig. 6 shows that the thermal boundary-layer thickness decreases for increasing values of Pr.

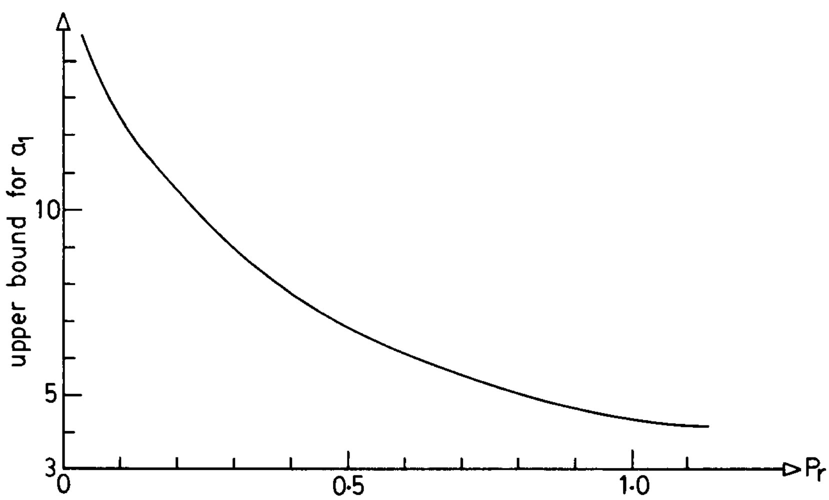

Fig. 5. The upper bound for a_1 as a function of Pr where $T_w = (x + b_2)/(0.4472a_1t + b_1)^2$.

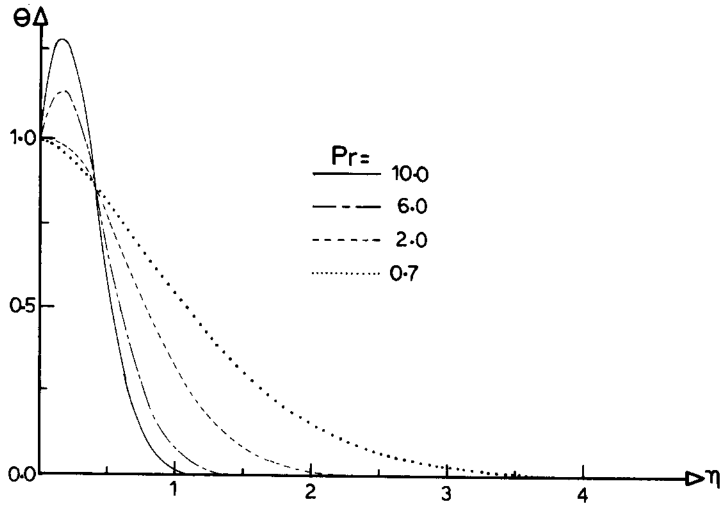

Fig. 6. Calculated dimensionless temperature profiles for fixed $T_w = (x + b_2)/(0.4472t + b_1)^2$ and varying values of Pr.

Figures 8 and 9 stand for the same study. The temperature and velocity profiles are obtained for the case $a_1 = 1$. This represents a case of surface temperature which varies linearly with x and inversely with time.

The temperature profiles in Fig. 8 show that θ becomes negative in a certain region of the boundary layer and for values of $\Pr \geqslant 2$. This phenomenon is known as temperature defect which was investigated by Kulkarni et al. [18]. Yang et al. [34] also noticed the occurrence of this phenomenon in their study of the free convection over an isothermal plate immersed in a nonisothermal medium.

This phenomenon is accompanied by a reverse in the direction of the velocity known as

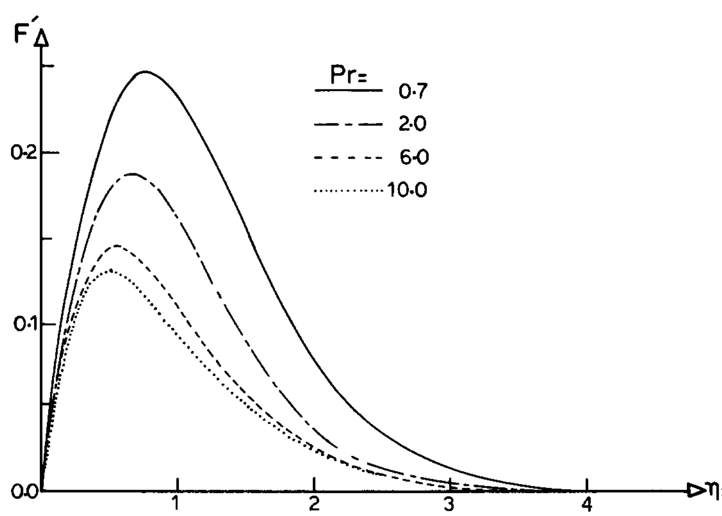

Fig. 7. Calculated dimensionless velocity profiles for fixed $T_w = (x + b_2)/(0.4472t + b_1)^2$ and varying values of Pr.

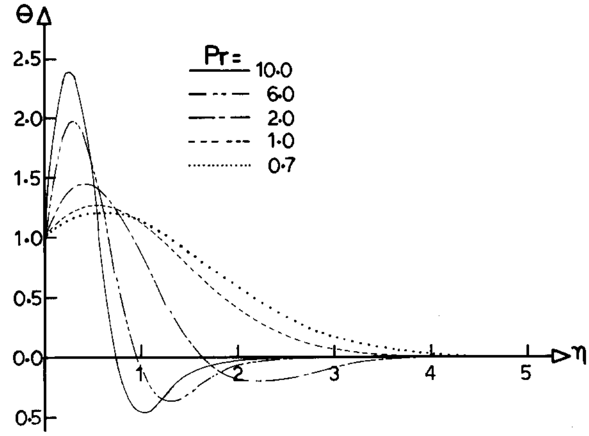

Fig. 8. Calculated dimensionless temperature profiles for fixed $T_w = (x + b_2)/(t + b_1)^2$ and varying values of Pr.

flow reversal, as can be seen in Fig. 9 for the profiles corresponding to Pr = 2, 6 and 10. It is shown that the flow reversal occurring in the profile corresponding to Pr = 2 is greater in magnitude than that corresponding to Pr = 6 and 10. This situation is reversed for the case of the temperature defect. It is also evident that both temperature and velocity do not exhibit any defect for the cases corresponding to values of Pr = 0.7 and 1.

Figures 8 and 9 also indicate the occurrence of this phenomenon nearer to the plate for larger Pr.

The validity of the relation between Pr and both the thickness of thermal boundary layer and the flow reversal, as has been noticed in Figs 8 and 9, can be extended to expect that the flow reversal will vanish for the limiting case $Pr \rightarrow \infty$, the case of viscous oil. By contrast, the reversal of temperature may be significant in this case. This behaviour may be consistent with the definition of Pr as a physical property of the fluid.

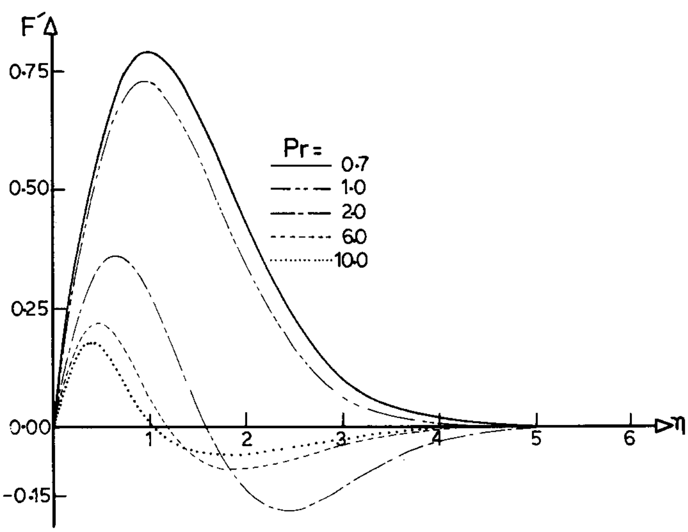

Fig. 9. Calculated dimensionless velocity profiles for fixed $T_w = (x + b_2)/(t + b_1)^2$ and varying values of Pr.

Special case of (2-b)

Figures 10 and 11 represent the temperature and velocity profiles corresponding to the case of linearly increasing surface temperature with time, i.e., $r = 1$. The results are obtained for $\mathrm{Pr} = 0.7$, 1, 2, 6 and 10.

The relation between Pr and the heat transfer at the surface is illustrated in Fig. 12. It shows that $-\theta'(0)$ increases with increasing Pr.

As it is given by equation (6.17), the surface temperature varies with time to the power r. A set of negative and positive values of r corresponds, respectively, to decrease and increase in T_w with time.

Figures 13 and 14 illustrate the temperature and velocity profiles corresponding to $r = -0.5$, 0.5, 1 and 2.

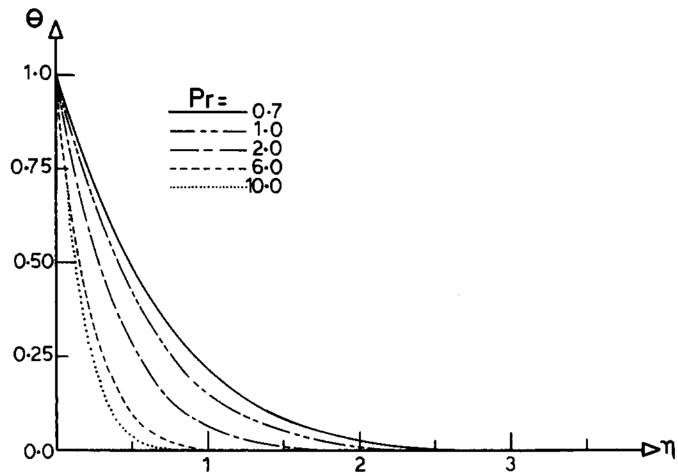

Fig. 10. Calculated dimensionless temperature profiles for fixed $T_w = t + b_1$ and varying values of Pr.

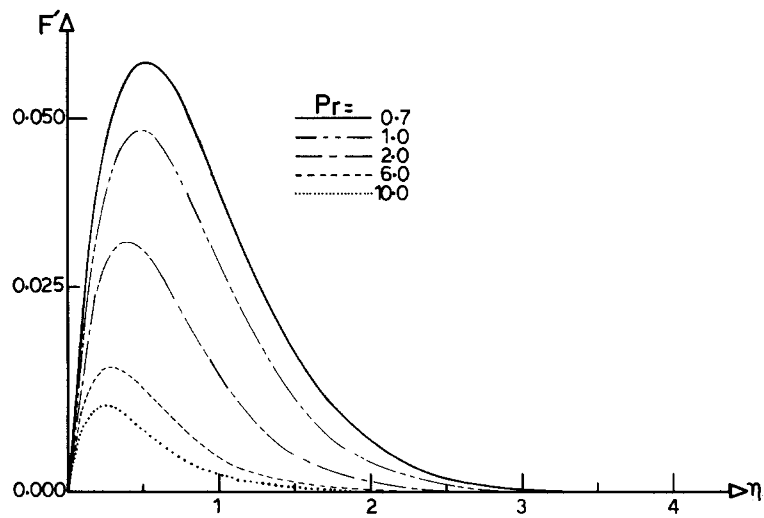

Fig. 11. Calculated dimensionless velocity profiles for fixed $T_w = t + b_1$ and varying values of Pr.

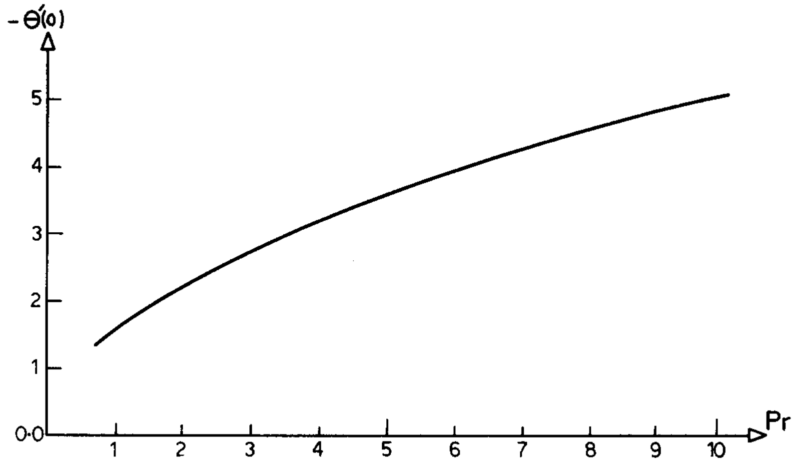

Fig. 12. Plot of the surface heat flux, $-\theta'(0)$ against Pr for $T_w = t + b_1$.

The effect of r on the heat transfer is investigated in Fig. 15 for a fixed $\mathrm{Pr} = 0.7$. As it is shown, the surface heat flux increases with r, i.e., for increasing T_w with time (positive values of r). The value of $-\theta'(0)$ is positive for all positve values of r. This situation is reversed when $r \leqslant -0.5$. The numerical computations show that no numerical solution can be obtained for cases for which $r < -1$; this is evident from Fig. 15 which shows an asymptotic tendency for $-\theta'(0)$ to infinity. Solution (6.26) for the special differential equation (6.25) shows also that solution does not exist for $r = -1, -2, -3, -4, \ldots$.

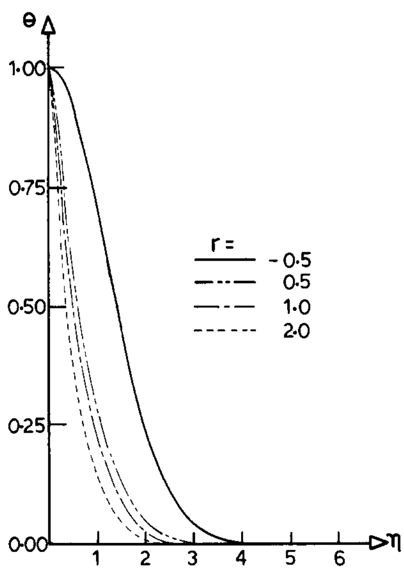

Fig. 13. Calculated dimensionless temperature profiles for fixed $\mathrm{Pr} = 0.7$ and varying values of $T_w = (t + b_1)^r$.

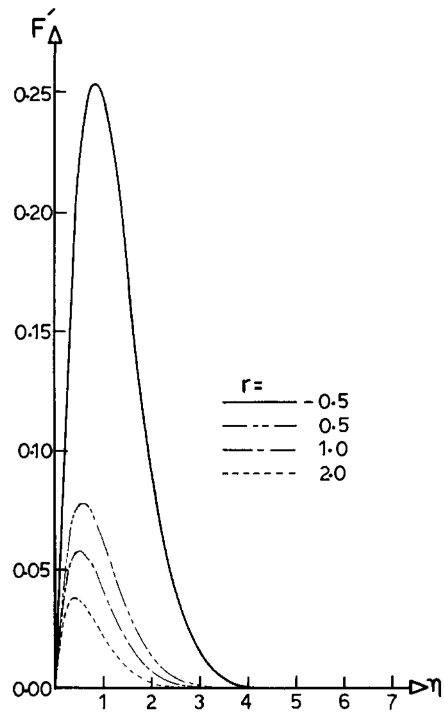

Fig. 14. Calculated dimensionless velocity profiles for fixed $\mathrm{Pr} = 0.7$ and varying values of $T_w = (t + b_1)^r$.

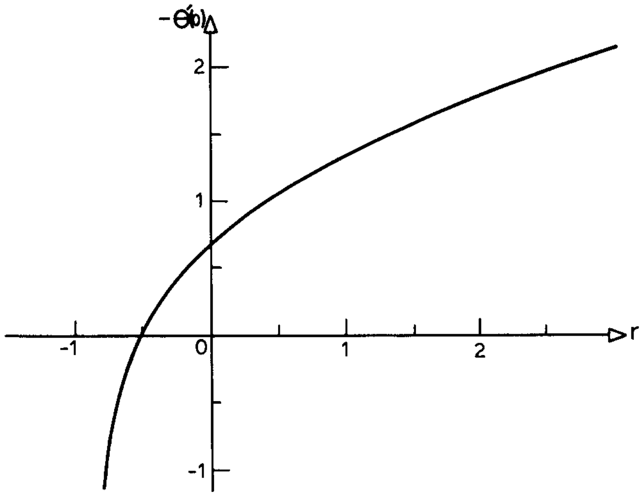

Fig. 15. Plot of the surface heat flux, $-\theta'(0)$, against r for fixed $\Pr = 0.7$ and $T_w = (t + b_1)^r$.

Acknowledgements

We acknowledge with thanks Mrs. Soraya Garas, The American University in Cairo, for her tireless efforts in typing the manuscript and for an excellent job. The authors would also like to thank the referee for suggesting certain changes in the original manuscript.

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